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Fractional derivative formulae in the form of difference operators

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Abstract

This paper presents interdisciplinary work between Fractional Calculus and Numerical Analysis. Authors established new formulae of Fractional derivative in the form of Forward and Backward Differences. Fractional derivatives of x^n, $\cos x$ and General Class of polynomial $S_n^m(x)$ with the help of newly defined formulae also obtained.

Key Words: Forward Difference Operator, Backward Difference Operator, Fractional Derivative, Hypergeometric Function.
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1. Introduction

1.1 Notations

Following notations used for deriving several results.
$\triangle_h$ = Forward Difference Operator, $\bigtriangledown_h$ = Backward Difference Operator, D = Differential Operator, E = Shift Operator, I = Identity Operator, h = Interval of Differences, $\mathbb{R}$ = Set of Real Numbers and $\mathbb{N}$ = Set of Natural Numbers.

1.2 Definitions

Let $t \in \mathbb{R}$ and $f(t)$ is a function of t then for $n \in \mathbb{R}$, following Operators defined as:
Shift Operator

$$E^{nh}f(t) = f(t+nh),\ E^{-jh}f(t) = f(t-jh)$$

Forward Difference Operator

$$\triangle_h f(t) = f(t+h) - f(t)$$

Backward Difference Operator

$$\bigtriangledown_h f(t) = f(t) - f(t-h)$$

Differential Coefficient

$$Df(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

1.3 Formulas

Well-known relationships between Shift Operator, Finite Differences and Differential Coefficient are given by

$$E^h \equiv e^{hD} \equiv I + \triangle_h \tag{1}$$

and

$$E^{-h} \equiv e^{-hD} \equiv I - \bigtriangledown_h \tag{2}$$

where $D \equiv \frac{1}{h}\left[\bigtriangledown_h + \frac{\bigtriangledown_h^2}{2} + \frac{\bigtriangledown_h^3}{3} - ....\right]$

$$Df(t) = f^{(1)}(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = \lim_{h\to 0}\frac{\triangle_h f(t)}{h} = \lim_{h\to 0}\frac{\bigtriangledown_h f(t+h)}{h} \tag{3}$$

for higher order

$$D^{(n)}f(t) = f^{(n)}(t) = \lim_{h\to 0}\frac{\triangle_h^n f(t)}{h^n} = \lim_{h\to 0}\frac{\bigtriangledown_h^n f(t+nh)}{h^n} \tag{4}$$

$$\bigtriangledown_h^n f(t) = (I - E^{-h})^n f(t) = \sum_{j=0}^{n}(-1)^j\, {}^nC_j E^{-jh} f(t) \tag{5}$$

$$\bigtriangledown_h^n f(t) = \sum_{j=0}^{n}(-1)^j\, {}^nC_j e^{-jhD} f(t) \tag{6}$$

$$\bigtriangledown_h^n f(t) = \sum_{j=0}^{n}(-1)^j\, {}^nC_j \sum_{i=0}^{\infty}\frac{(-hjD)^i}{(i)!} f(t) \tag{7}$$

Formula for fractional order differences (CISM Lecture Notes [3]) defined as

$$\bigtriangledown_h^\alpha f(t) = \sum_{j=0}^{\infty}(-1)^j\, {}^\alpha C_j E^{-jh} f(t) \tag{8}$$

$$\bigtriangledown_h^\alpha f(t) = \sum_{j=0}^{\infty}(-1)^j\, {}^\alpha C_j e^{-jhD} f(t) \tag{9}$$

$$\nabla_h^\alpha f(t) = \sum_{j=0}^{\infty} (-1)^j \, {}^\alpha C_j \sum_{i=0}^{\infty} \frac{(-hjD)^i}{(i)!} f(t) \tag{10}$$

2. Main results

Result 1. The fractional forward and backward differences formula in terms of Derivatives for $\alpha \in \mathbb{R}^+$

$$\triangle_h^\alpha f(t) = (hD)^\alpha \sum_{j=0}^{\infty} {}^\alpha C_j \left(\frac{hD}{2!} + \frac{h^2D^2}{3!} + ... \right)^j f(t) \tag{11}$$

and

$$\nabla_h^\alpha f(t) = (hD)^\alpha \sum_{j=0}^{\infty} {}^\alpha C_j \left(\frac{hD}{2!} - \frac{h^2D^2}{3!} + ... \right)^j f(t) \tag{12}$$

Proof.
For forward difference,
from equation (1), we have

$$\begin{aligned} \triangle_h f(t) &= (e^{hD} - I) f(t) \\ &= hD \left[1 + \frac{hD}{2!} + \frac{h^2D^2}{3!} + ... \right] f(t) \end{aligned}$$

for n^{th} difference, we get

$$\begin{aligned} \triangle_h^n f(t) &= h^n D^n \left[1 + \left(\frac{hD}{2!} + \frac{h^2D^2}{3!} + ... \right) \right]^n f(t) \\ &= h^n D^n \sum_{j=0}^{n} {}^n C_j \left(\frac{hD}{2!} + \frac{h^2D^2}{3!} + ... \right)^j f(t) \end{aligned}$$

this formula can be generalized for fractional order differences (CISM Lecture Notes [3]) as

$$\triangle_h^\alpha f(t) = h^\alpha D^\alpha \sum_{j=0}^{\infty} {}^\alpha C_j \left(\frac{hD}{2!} + \frac{h^2D^2}{3!} + ... \right)^j f(t).$$

Similarly, for backward difference,
from equation (2), we have

$$\begin{aligned} \nabla_h f(t) &= (I - e^{-hD}) f(t) \\ &= hD \left[1 - \frac{hD}{2!} + \frac{h^2D^2}{3!} - .... \right] f(t) \end{aligned}$$

n^{th} difference gives

$$\begin{aligned}\nabla_h^n f(t) &= h^n D^n \left[1 - \left(\frac{hD}{2!} - \frac{h^2D^2}{3!} + ...\right)\right]^n f(t) \\ &= h^n D^n \sum_{j=0}^{n} {}^nC_j \left(\frac{hD}{2!} - \frac{h^2D^2}{3!} + ...\right)^j f(t)\end{aligned}$$

this formula can be generalized for fractional order differences (CISM Lecture Notes [3]) as

$$\nabla_h^\alpha f(t) = h^\alpha D^\alpha \sum_{j=0}^{\infty} {}^\alpha C_j \left(\frac{hD}{2!} - \frac{h^2D^2}{3!} + ...\right)^j f(t).$$

Result 2. Fractional Derivative formula in terms of Forward and Backward Differences are

$$D^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} {}^\alpha C_j (-1)^j \sum_{i=0}^{\infty} {}^{-j}C_i (-1)^i \triangle_h^i f(t+\alpha h) \tag{13}$$

$$D^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} {}^\alpha C_j (-1)^j \sum_{i=0}^{\infty} {}^{j}C_i (-1)^i \nabla_h^i f(t+\alpha h) \tag{14}$$

another forms

$$D^\alpha f(t) = \frac{\triangle_h^\alpha}{h^\alpha} \sum_{j=0}^{\infty} {}^\alpha C_j (-1)^j (\frac{\triangle_h}{2} - \frac{\triangle_h^2}{3} + ...)^j f(t) \tag{15}$$

$$D^\alpha f(t) = \frac{\nabla_h^\alpha}{h^\alpha} (-1)^{2\alpha} \sum_{j=0}^{\infty} {}^\alpha C_j (\frac{\nabla_h}{2} + \frac{\nabla_h^2}{3} + ...)^j f(t) \tag{16}$$

Proof From equations (4) and (5), we have

$$D^n f(t) = \lim_{h\to 0} \frac{1}{h^n} \sum_{j=0}^{n} {}^nC_j (-1)^j E^{-jh} f(t+nh)$$

this formula can be generalized for fractional order derivatives (CISM Lecture Notes [3]) as

$$D^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} {}^\alpha C_j (-1)^j E^{-jh} f(t+\alpha h) \tag{17}$$

from (1), we have

$$E^{-jh} \equiv (I + \triangle_h)^{-j} \equiv \sum_{i=0}^{\infty} {}^{-j}C_i \triangle_h^i \tag{18}$$

and

$$E^{-jh} \equiv (I - \bigtriangledown_h)^{j} \equiv \sum_{i=0}^{\infty} {}^{j}C_i (-1)^i \bigtriangledown_h^i \tag{19}$$

from (17) and (18), we get

$$D^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} {}^{\alpha}C_j (-1)^j \sum_{i=0}^{\infty} {}^{-j}C_i (-1)^i \triangle_h^i f(t + \alpha h)$$

This completes the proof of (13).
Equations (17) and (19) leads to

$$D^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} {}^{\alpha}C_j (-1)^j \sum_{i=0}^{\infty} {}^{j}C_i (-1)^i \bigtriangledown_h^i f(t + \alpha h)$$

This gives (14).
Again from (1), we have

$$\begin{aligned} D^n f(t) &\equiv \frac{1}{h^n} \left[\triangle_h - \frac{\triangle_h^2}{2} + \frac{\triangle_h^3}{3} - \ldots\ldots \right]^n f(t) \\ &\equiv \frac{\triangle_h^n}{h^n} \left[1 - \left(\frac{\triangle_h}{2} - \frac{\triangle_h^2}{3} + \ldots\ldots \right) \right]^n f(t) \end{aligned}$$

using Binomial expansion, we obtain

$$D^n f(t) = \frac{\triangle_h^n}{h^n} \sum_{j=0}^{n} {}^{n}C_j (-1)^j \left(\frac{\triangle_h}{2} - \frac{\triangle_h^2}{3} + \ldots\ldots\ldots \right)^j f(t),$$

this formula can be generalized for fractional order derivatives (CISM Lecture Notes [3]) as

$$D^{\alpha} f(t) = \frac{\triangle_h^{\alpha}}{h^{\alpha}} \sum_{j=0}^{\infty} {}^{\alpha}C_j (-1)^j \left(\frac{\triangle_h}{2} - \frac{\triangle_h^2}{3} + \ldots\ldots\ldots \right)^j f(t).$$

This leads the proof of (15).
Again from (2), we have

$$D^n f(t) \equiv \frac{(-1)^n}{h^n} \left[- \bigtriangledown_h - \frac{\bigtriangledown_h^2}{2} - \frac{\bigtriangledown_h^3}{3} - \ldots\ldots \right]^n f(t)$$

$$D^n f(t) \equiv \frac{(-1)^{2n}\nabla_h^n}{h^n}\left[1+\left(\frac{\nabla_h}{2}+\frac{\nabla_h^2}{3}+.......\right)\right]^n f(t)$$

using Binomial expansion, we obtain

$$D^n f(t) = \frac{(-1)^{2n}\nabla_h^n}{h^n}\sum_{j=0}^{n} {}^nC_j\left(\frac{\nabla_h}{2}+\frac{\nabla_h^2}{3}+.........\right)^j f(t),$$

this formula can be generalized for fractional order derivatives (CISM Lecture Notes [3]) as

$$D^\alpha f(t) = \frac{(-1)^{2\alpha}\nabla_h^\alpha}{h^\alpha}\sum_{j=0}^{\infty} {}^\alpha C_j\left(\frac{\nabla_h}{2}+\frac{\nabla_h^2}{3}+.........\right)^j f(t).$$

this follows the proof of (16).

Result 3. The Fractional derivative of x^n is given by

$$D^\alpha(x^n) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)_j(x+\alpha h)^n}{(j)!}\ {}_1F_0\left[-n;-;\frac{jh}{x+\alpha h}\right] \tag{20}$$

where $\alpha \leq n$.

Proof From equation (17), we have

$$D^\alpha(x^n) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{j=0}^{\infty} {}^\alpha C_j(-1)^j E^{-jh}(x+\alpha h)^n, \tag{21}$$

using (2), we obtain

$$\begin{aligned} D^\alpha(x^n) &= \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{j=0}^{\infty} {}^\alpha C_j(-1)^j e^{-jhD}(x+\alpha h)^n \\ &= \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{j=0}^{\infty} {}^\alpha C_j(-1)^j \sum_{i=0}^{\infty}\frac{(-jhD)^i}{(i)!}(x+\alpha h)^n \end{aligned}$$

this equation reduces to,

$$D^\alpha(x^n) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{j=0}^{\infty} {}^\alpha C_j(-1)^j \sum_{i=0}^{\infty}\frac{(-1)^j(hj)^i(n)!}{(i)!(n-i)!}(x+\alpha h)^{n-i} \tag{22}$$

The following result (23) mentioned in (Erdelyi et al [1], page 85)

$$ {}^{\alpha}C_j = \frac{(-1)^j \Gamma(j-\alpha)}{\Gamma(j+1)\Gamma(-\alpha)} \tag{23}$$

From (22) and (23), we obtain

$$D^{\alpha}(x^n) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} \frac{(-\alpha)_j (x+\alpha h)^n}{(j)!} \sum_{i=0}^{\infty} \frac{(-n)_i}{(i)!} \left(\frac{jh}{x+\alpha h}\right)^i,$$

$$= \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} \frac{(-\alpha)_j (x+\alpha h)^n}{(j)!} \, {}_1F_0 \left[-n; -; \frac{jh}{x+\alpha h}\right]$$

Result 4. The Fractional derivative of $\cos x$ is given by

$$D^{\alpha} \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} \frac{(-\alpha)_j (x+\alpha h)^{2k}}{(j)!} \, {}_1F_0 \left[-2k; -; \frac{jh}{x+\alpha h}\right] \tag{24}$$

Result 5. The Fractional derivative of $S_n^m(x)$ a general class of polynomial is given by

$$D^{\alpha} S_n^m(x) = \sum_{k=0}^{\frac{n}{m}} \frac{(-n)_{mk}}{(k)!} A_{m,k} \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} \frac{(-\alpha)_j (x+\alpha h)^k}{(j)!} \, {}_1F_0 \left[-k; -; \frac{jh}{x+\alpha h}\right] \tag{25}$$

Proof. The general class of polynomial given by (Srivastava [2]), is

$$S_n^m(x) = \sum_{k=0}^{\frac{n}{m}} \frac{(-n)_{mk}}{(k)!} A_{m,k} \, x^k \tag{26}$$

where m is the arbitrary positive integer, the coefficient $A_{m,k}; n, k > 0$ are arbitrary constant real or complex.
Using the result of theorem 2 and equation (26), we immediately get the desire result.

Note: Thus, we can easily obtain fractional derivatives of all functions and polynomials which are in power series forms.

References

[1] Erdelyi, A. et al. Higher Transcendental Function, Vol 1, McGraw- Hill Book Company, INC., 1953.

[2] Srivastava, H.M., A contour integral involving Fox's H-function, Indian J. Math. 14 (1972), 1-6.

[3] CISM Lecturer Notes, Published in Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien and New York 1997, PP 277-290.